University of Illinois at Urbana-Champaign Department of Physics

Physics 401 Classical Physics Laboratory

Torsional Oscillator

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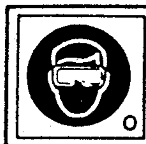


SAFETY GLASSES ARE OPTIONAL (AND AVAILABLE).

I. Introduction

The goals of this experiment are threefold: first, to study the transient motion of a mechanical oscillator; second, to study forms of dissipation besides viscous damping; and third, to study the behavior of a driven oscillator.

A damping force that is proportional to the velocity (linear damping) is made by the eddy currents induced by the motion of the oscillator in a magnetic field. This magnetic damping is analogous to the effect of the resistance in an RLC circuit. Coulomb (or dry) friction is another form of damping that can occur in mechanical systems. Coulomb friction, since it is independent of velocity, will exhibit a different form of damping. Finally, turbulent dissipation can occur in mechanical systems. Turbulent friction is found in the motion of air around a fast moving car or in the motion of water around a boat. Such dissipation can increase as the square (or larger) power of velocity. Turbulent friction also will exhibit a different form of damping. The apparatus is illustrated in Fig. 1.

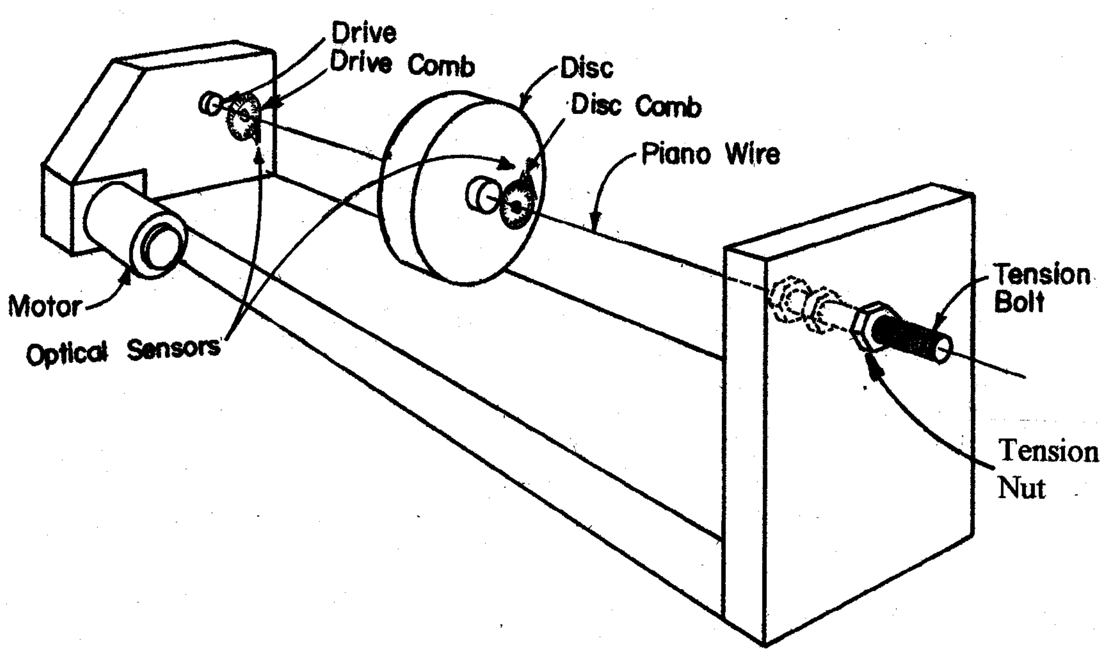


Fig. 1 Optical sensors measure the position of the disc and the drive. One sensor is mounted on the side of the disc. The second sensor is mounted on the piano wire at the drive end.

II. Experimental Setup

This experiment will be carried out with a 4 inch diameter copper disc fastened to the center of a piano wire as shown in Fig. 1. Before beginning the exercises note that the large magnet should be kept as far away from the computer as possible. The large field could affect the computer's hard drive. Your wrist watch and credit cards should also be kept far from the magnet.

The positions of the opto-electronic sensors relative to the code wheels attached to the disc and drive motor have been adjusted with an oscilloscope for optimal operation. The positioning screws have been covered with red tape. Do not remove the tape and do not adjust the positioning screws. The code wheels are very delicate high angular resolution devices and can be easily damaged. Therefore, whenever you are moving the torsional pendulum by hand, or positioning the magnet be extremely careful not to bump or touch either of the code wheels.

The 16-bit instantaneous angular position data associated with the torsional pendulum and motor code wheels is stored locally in memory on a front-end electronics buffer board, which is read out via the TORSNHP6 LabWindows/CVI program. The front-end electronics buffer board insures that the code wheel data is captured at uniform time intervals. The time interval, or data acquisition rate, is set in hardware on the buffer board via a binary code implemented using four DIP switches. The default data acquisition rate is 50 Hz, which corresponds to a time interval of 20 msec. This sampling rate optimizes the resolution on the angular velocities associated with the torsional pendulum apparatus. See the appendix for the position of the DIP switches to achieve a given data acquisition rate, varying from 5Hz to 200Hz. If the hardware data acquisition rate is changed from its default setting, then the TORSIONHP6 program must also be informed of this change in data acquisition rate from the nominal/default setting, via the program's Time Base pull-down menu. If the hardware setting of the sampling time on the buffer board vs. what the TORSNHP6 program thinks it is, then the program will not analyze the code-wheel data correctly.

The data acquisition electronics must be first initialized using the mouse to click on the INIT_DAQ button on the TORSNHP6 program. A DAQ Initialized LED turns red after initialization. Next, with both the torsional pendulum and motor motionless, click on the ZERO_Counters button to zero the code-wheel up/down counters, thus defining the angular equilibrium/zero positions for each {note: when the motor is then turned on, unless the motor is at its true equilibrium position, the equilibrium positions for both torsional pendulum and motor will be changed – post-facto, note that you can also use the SHIFT_DATA button to correct for a non-zero offset in the torsional pendulum angular position data}. Use the mouse to click on the START button to initiate data-taking. When the STOP button is pressed, the pendulum/motor code-wheel data is then read out from the front-end electronics buffer board associated with the STOP-START time interval. The TORSNHP6 program will plot the torsional pendulum/motor angular position, velocity and acceleration data on-line via pull-down menus and also allows a certain level of data analysis on-line. The raw angular position data is simply the code-wheel sensor reading versus sample number. One revolution of the disc (2π radians) equals a code-wheel sensor reading of 8192. The pendulum/motor angular velocities and accelerations are calculated using finite time differences of angular position data. The TORSNHP6 program has a pull-down menu offering a choice of four algorithms for calculating angular velocity and acceleration. The first choice is the simplest – using finite differences associated with nearest-neighbor data (n and n-1 elements). The second algorithm choice is similar, but uses (n+1, n and n-1 elements). The third algorithm uses a weighted average of 75% from symmetric nearest neighbors (n+1, n and n-1 elements) and 25% from next-to-nearest neighbors (n+2 and n-2). The fourth algorithm, which is the default/nominal algorithm uses a more sophisticated approach – known as the Verlet integration method, which uses leading-order terms associated with the Taylor series expansion of the angular position finite-difference data. We encourage you to investigate/compare velocity/acceleration data calculated via each of these four algorithms. The WRITE_DATA button writes out the torsional pendulum and motor data to the disk. The position of the disc at the moment when data acquisition begins is taken to be the zero position. This offset can be corrected for in Origin when you analyze your data.

1. How to use the drive motor:

The drive motor applies torque to the torsional pendulum through a series of small steps. A pulse applied to the motor rotates the shaft by a small and well defined angle. By controlling the rate of the applied pulses, the angular velocity of the motor can be accurately controlled. This type of motor is called a stepper motor.

A stepper motor controller receives pulses from an external pulse generator and produces a current pulse that is applied to the stepper motor. In our experiment, we will use the Wavetek Function Generator to supply the pulses to the setter motor controller. The drive frequency is set by changing the frequency of the Wavetek. To determine the drive frequency, connect the CAL OUT connector of the controller to the HP Digital Multimeter using the BNC to banana connector. The multimeter should be set to FREQ mode. The drive frequency is given by

$$\text{drive frequency in Hz} = \frac{\text{CAL OUTPUT in Hz}}{1{,}600},$$

or

$$\text{drive angular frequency in rad/s} = \frac{\text{CAL OUTPUT in Hz}}{254.6}$$

On the frequency counter, push the "atten" button in, and the "filt" button out. The frequency reading should be very stable. When setting the frequency settings of the Wavetek, the stepper motor controller should be in the STOP position. The function generator should be set to positive square wave output, $V_{out} = 2.50\ V_{pp}$, and $V_{offset} = 0$ V. Set the frequency to an initial value of 300 Hz. To turn the motor on, set the controller to the RUN position.

The mechanism that connects the drive motor to the wire is covered for safety concerns. The motor rotates at a constant rate, and the connection mechanism causes the wire to oscillate approximately sinusoidally; the linkage between the drive motor and the wire does not produce perfect sinusoidal motion. There are small components in the torque on the wire at twice and three times the drive frequency. The presence of these components can be found by running the drive motor at one-half or one-third of the resonant frequency and observing that the steady state motion of the disk is not purely sinusoidal.

There may be a background sound, like grinding metal, which is normal. It is not destroying itself. However, if the motor stalls by being started or run at too high a speed, it should be stopped, and its speed reduced before restarting. In general, you should turn on the motor at a low speed and then raise the speed.

III. Exercises

WARNING: Be very careful when driving the oscillator near resonance. The piano wire can break, which is very dangerous. ALWAYS keep the plastic covering on when operating the oscillator, regardless of the frequency, for your safety.

This is a two-week experiment. It is suggested that you spend the first week measuring various sources of damping and the second week using the drive motor; however, you may proceed through the experiment in whatever fashion you see fit. Immediately following this section are some experiments. Following the experiments is a rather detailed section on data analysis, including Fourier transformations. Regardless of what experiments you choose do, it is expected that you will perform this type of data analysis.

Exercise 1. Determine the torsional constant from static measurements.

The masses and dimensions of the copper disc and plastic disc are written on the discs. Calculate the moment of inertia of the disc. Measure the diameter of the piano wire with the precision micrometer, using any necessary technique to minimize the uncertainty. Measure and record the lengths of wire, L_1 and L_2 (see Fig. 2 in Appendix I) which support the disc. To measure the torsional spring constant of the piano wire, you will use a static displacement of the disc under a known torque. The method is illustrated in Fig. 4.

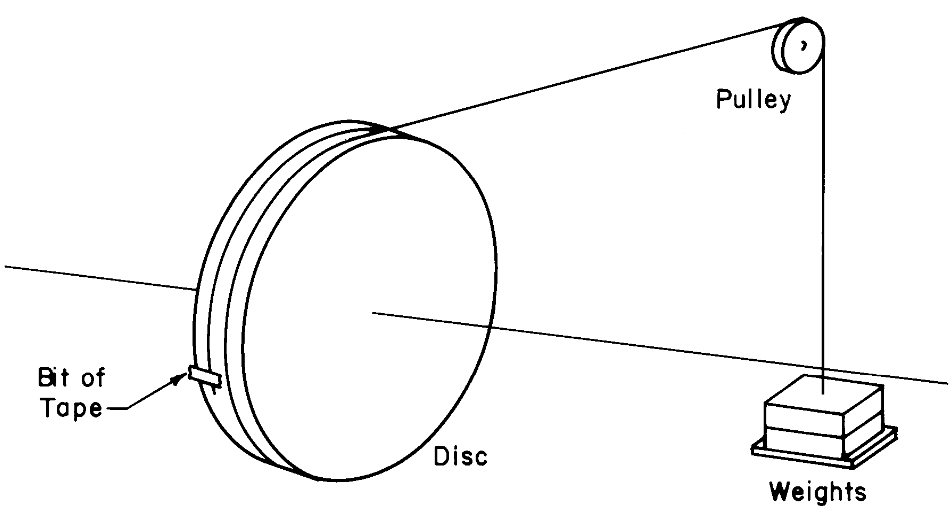


Figure 4 Measurement of Torsional Spring Constant

You will need to clamp a pulley to the edge of the table and use steps of 20 grams to crank the disc. In order to stabilize the disc of the torsional pendulum quickly, position the magnet around the disc to help damp its motion. Start the TORSNHP6 program and add one mass at a time until you reach 100g, then remove them one at a time.

The spring constant *K* is determined from the slope of the line of angle θ versus mass *m*

$$\theta = \frac{g\rho}{K} m . \tag{1}$$

In Eq. 1 *g* is the gravitational acceleration of 9.80 m/sec^2 and ρ is the radius of the disc. Uncertainty in the mass may be taken to be ±0.01 g. Determine the torsional constant, *K*, and its uncertainty by analyzing these data in Origin. Sample data are shown in figure 5.

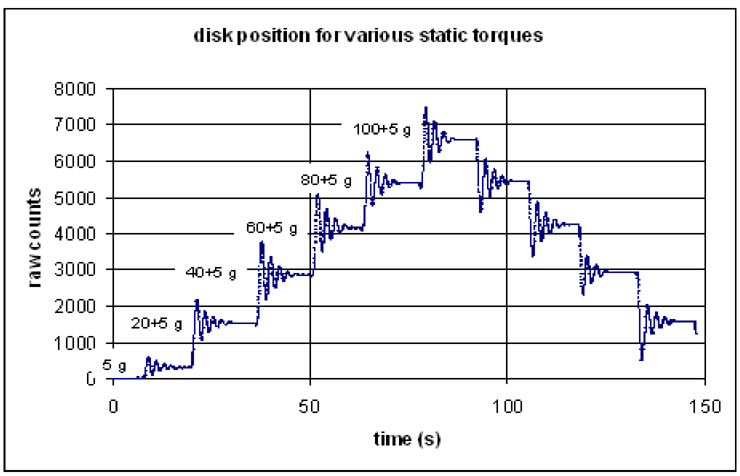


Fig. 5 Disk position for various static torques

Exercise 2. Determine the torsional constant from dynamic measurements.

Carefully measure the natural frequency of the copper disc. From your data, calculate the natural frequency. Since $\omega_o = \sqrt{K/I}$, you can compare this value of K with the value you found earlier. You can also plot angular acceleration versus position to calculate K. Sample data are shown in figure 6.

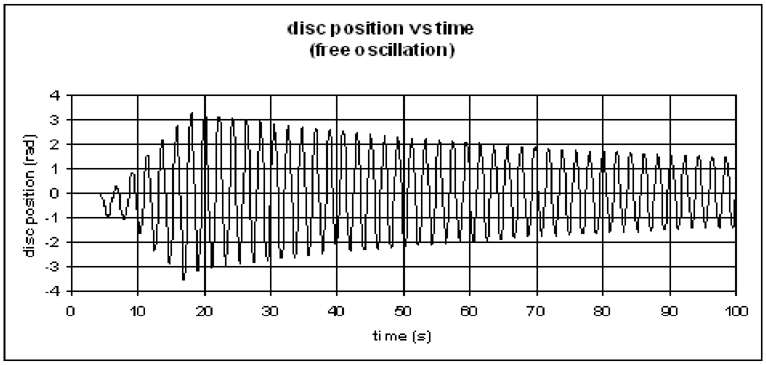


Fig. 6 Disc position versus time for undamped oscillation

Exercise 3. Under- and over-damped motion with viscous damping

Viscous damping can be achieved by placing the disc between the poles of a magnet. The eddy currents induced in the disc by its motion are proportional the angular velocity of the disc, and these eddy currents will exponentially damp the motion of the disc. The exact equations for viscous damping are discussed in Appendix I. Move the magnet poles to completely surround the copper disc. The motion of the disc should be "over damped," i.e. decay with no oscillation. Record this motion. Next, back the magnet away from the disc slightly and repeat until you observe underdamped motion. You may also try to find the critically damped position though this may prove difficult. Sample data are shown in figure 8.

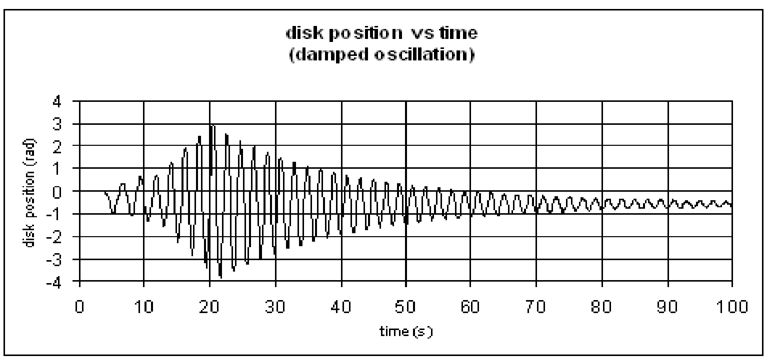


Fig. 8 Disc position versus time for viscous under-damped motion

Exercise 4. Demonstrate Coulomb (kinetic frictional) damping

The motion of the disc can be damped by kinetic friction, also known as Coulomb damping. Coulomb (frictional) damping is independent of the magnitude but dependent on the direction of the velocity. This type of damping will cause the amplitude to decrease linearly with time. To study Coulomb damping remove the magnet and position the bristles of the small paint brush close to the rim of the disc. Sample data are shown in figure 9.

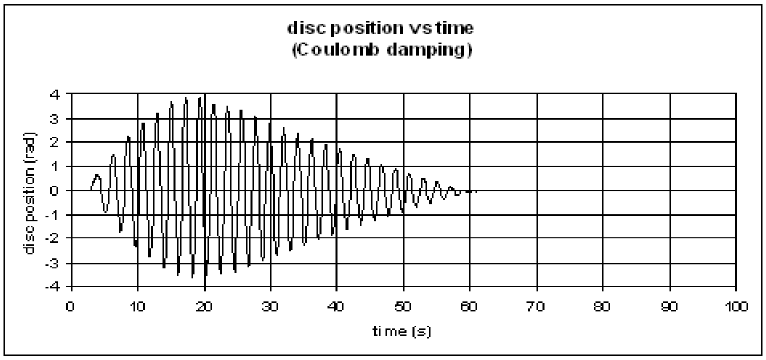


Fig. 9 Disc position versus time for Coulomb damped motion

Exercise 5. Demonstrate turbulent damping

The "undamped" oscillator is, of course, subject to damping mechanism. These damping mechanisms are present with the magnet and with the paint brush, but they are negligible compared to the magnetic or the Coulomb damping. One mechanism is drag from the surrounding air. We must increase the drag to demonstrate turbulent damping. The addition of light-weight vanes to the disc will increase the drag from the air and produce turbulent damping. Attach either the cardboard or styrofoam pieces perpendicular to the disk with tape, then observe the oscillations. Try to maintain the symmetry of the disk when placing the vanes. You will need to take data for a very long time to see this effect.

This is the end of the experiments for the first week. We will now move on to experiments involving the drive motor.

Exercise 6. Demonstrate the phenomenon of beats

The superposition of the undamped transient motion and the driven motion can be arranged to demonstrate the phenomenon of beats. Since damping will cause the transient to die away, the magnet should be far away from the disk so that the motion is undamped. You can recall from the discussion above that the motor control box generates a signal, CAL OUTPUT, which is proportional to the drive frequency,

$$\textit{drive frequency in Hz} = \frac{\textit{CAL OUTPUT in Hz}}{1,600},$$

You should calculate the CAL OUTPUT in Hz which corresponds to the natural frequency of the disk. Note that the frequency counter displays in kHz. Choose a drive frequency just below resonance, for example $f \approx 0.9 f_o$, to observe the beats. **Do not try to set the drive exactly to resonance. Without damping, the amplitude of the oscillation can become dangerously large, causing the piano wire to break.** Sample data are shown in figure 10. It can be somewhat challenging to produce a good beat pattern.

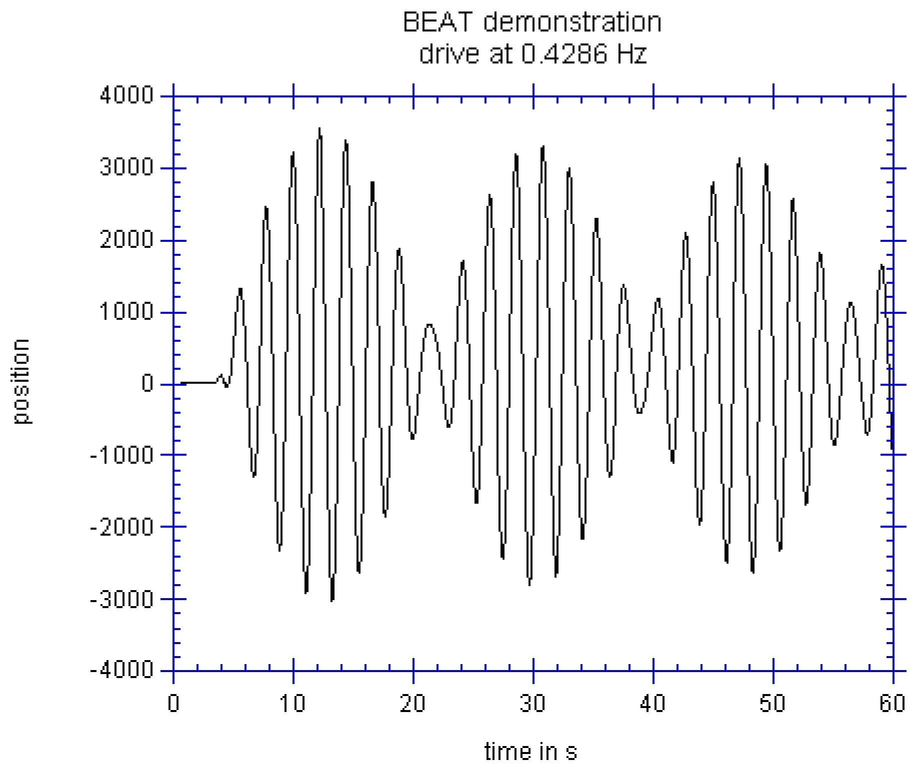


Fig. 10 Beat frequency demonstration (real data)

Exercise 7. Measure the amplitude and phase of damped, driven oscillator

You now move on to the major part of the laboratory exercise: the measurement of the amplitude and the phase of damped, driven oscillator as you vary the driving frequency. Recall the expressions

$$\theta(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right) \quad B(\omega) = \frac{\lambda\theta_o\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}} \quad \tan\beta(\omega) = \frac{\omega\gamma}{\left(\omega_o^2 - \omega^2\right)},$$

for the steady state motion with amplitude $B(\omega)$ and phase $\beta(\omega)$. The natural frequency of the disk is fixed. It is about 0.5 Hz. The drive amplitude is fixed. It is about 0.65 rad. You do have control over the damping parameter. If you would like to have a large change in the amplitude of the disk near resonance, you must choose small damping. Fig. 11 below shows a damped oscillation. When the magnet is close to the disk, you have large damping. When it is far from the disk, you have small damping. You should try to find the magnet position which produces a damping time of about 5-10 s. Recall that the damping time is the time over which the amplitude of the oscillation decays by a factor of 1 / e = 0.368. The damped oscillation in Fig. 11 has a decay time between 7 and 8 seconds. Position the magnet a centimeter from the disk, and measure the decay time of the free oscillation. Observe the free oscillation data and adjust the magnet position accordingly. Record the decay time. After you have found the correct magnet position, you **must not move the magnet** for the entire series of measurements below. Moving the magnet changes the width of the resonance curve.

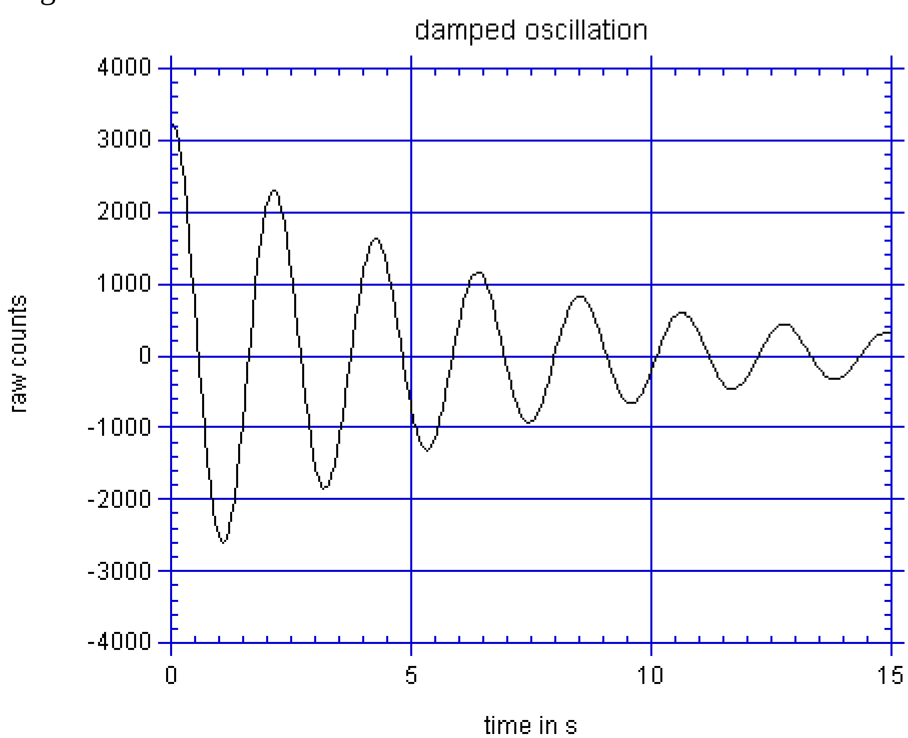


Fig. 11 Magnetically Damped Oscillation

Before beginning the measurement, look at Fig. 12. You can see that you must choose the drive frequencies carefully to see the resonance peak. For an oscillator with small damping, you could easily miss the resonance peak if your frequency steps are too large. You should plot amplitude and phase versus frequency as you take the data in order that you don't miss the peak. Remember that after each change of the drive frequency, the transient motion of the oscillator is excited. You must wait at least three decay times for this transient to decay, i.e. 15 to 30 seconds.

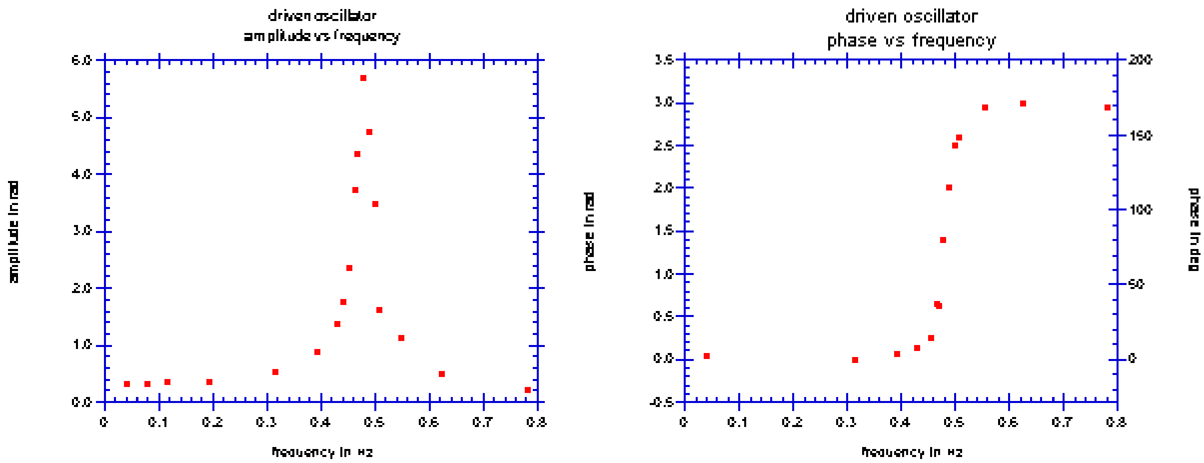


Fig. 12 Amplitude and phase of damped, driven oscillator versus frequency (real data)

Figure 13 below shows the motion of the oscillator and the drive motor as a function of time at a drive frequency of 0.477 Hz as read from the frequency counter. The peak-to-peak amplitude is indicated in the figure. Use the cursors to obtain this quantity. When there are two plots on the screen at once, the two cursors can cause confusion. When selecting two plots, you will see this notice on the screen:

> **For these plots the user must physically place the YELLOW cursor on the BLUE (Pendulum Data) curve and place the MAGENTA cursor on the MAGENTA (Motor Data) curve. Only then will X-Y cursor port data be correct.**

Fig. 13 also shows the time difference between the maximum amplitude of the drive motor and the maximum amplitude of the oscillator. This time difference is denoted as Δt in the figure. You can convert this time difference into a phase. Recall that a time difference of one period ($T = 1/f$) is a phase difference of 2π. Alternatively, you can download the time traces into Origin and fit the data to a sinusoidal function with a phase offset. This would be a more accurate determination of the amplitude, frequency and phase of the drive and the response. The phase difference between the drive and response can thus be determined.

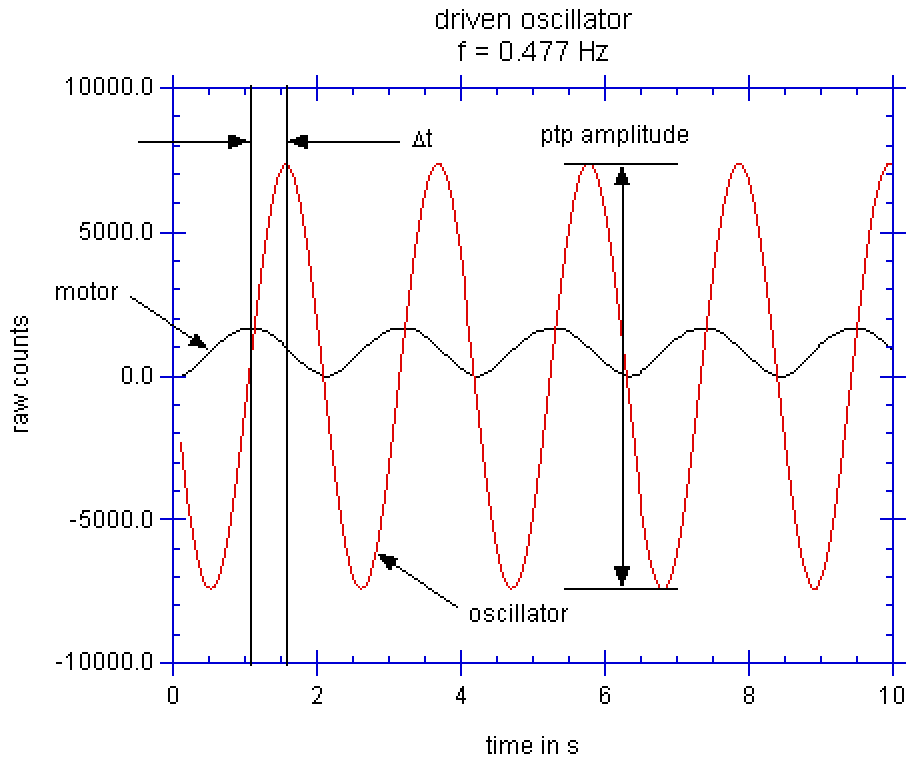


Fig. 13 Oscillator and drive versus time (real data)

Exercise 8. Drive the oscillator at a sub harmonic of the resonant frequency

Due to the geometry of the drive linkage mechanism, the drive motion is not purely sinusoidal! Empirically,

$$\theta_{drive}(t) = \theta_o\left(\cos\omega t + c_2\cos\left(2\omega t + \phi_2\right) + c_3\cos\left(3\omega t + \phi_3\right)\right)$$

where the amplitude of the second harmonic is about 6% and the amplitude of the third harmonic is about 3%. (The higher order terms are due to the simple linkage between the stepping motor and the piano wire. With the motor stopped, you may take off the cover plate and take a look.) If the drive frequency is set near a sub harmonic of the resonant frequency, the distortion gets magnified. This effect is further exacerbated by small damping. Set the drive frequency to a sub-harmonic and take some data. Plot the data and identify the harmonics in the motion of the disk.

Your report should address the following points:

1. Calculate the moment of inertia of the torsional oscillator. Argue quantitatively as to why the hub and plastic disc may be neglected in this calculation.

2. What are the values of the torsional constant from exercises 1 and 2? From your measurements of the torsional constant and the radius of the piano wire, calculate the shear modulus of the material from which the piano wire was made. Does your value of the shear modulus agree with the handbook value?

3. From the moment of inertia obtained in part 1 and the torsional constant obtained in part 2 above, calculate the natural frequency of the torsional oscillator. Remember to carry along errors for your error analysis. Compare the calculated natural frequency to the measured frequency. Include a plot of the measured oscillator motion.

4. Compare the motion of the disc for the two (or three) forms of dissipation you demonstrated with the data graphed in an appropriate way. Note differences in the motion in your discussion.

(a) For magnetic damping, calculate the attenuation constant, a, the logarithmic decrement, δ, and the quality factor, Q for the under damped motion.

(b) For Coulomb damping, find the frequency of the oscillation and compare it to the natural frequency. Determine the Coulomb damping constant from the slope of the maximum amplitude versus time graph.

(c) For turbulent damping plot the log decrement vs amplitude. Determine the region over which the dependence of log decrement on amplitude is linear. Over this region fit the data to a straight line which passes through the origin. Find the trubulent damping coefficient.

These questions pertain to the driven oscillator:

5. Compare the measured beat frequency to the expected beat frequency. Include a plot showing the beat frequency. See Figure 10.

6. Find the decay time of the magnetically damped motion. Use the semi-log plot of amplitude versus time. From the slope, determine the decay constant, *a*, the decay time, 1/*a*, the logarithmic decrement, δ, and the *Q* of the oscillator.

7. Plot the amplitude of the oscillator versus frequency. Compare the frequency at which the amplitude is a maximum to the natural frequency. Find the frequencies above and below the resonant frequency at which the amplitude is $1/\sqrt{2}$ of its peak value. The difference of these frequencies is the width of the resonance, γ. Recall the relation between the decay constant, *a*, and the resonance width for a magnetically damped oscillator, compare *a* to γ.

8. Plot the phase versus frequency. Find the frequency at which the phase is π/2. Compare this frequency to expectations.

9. Plot the motion of the disc when the drive frequency is a sub harmonic of the natural frequency. Identify the harmonics in the disc motion.

Appendix I

II. Theory

We will extend the theory presented in the RLC circuit lab to the torsional oscillator.

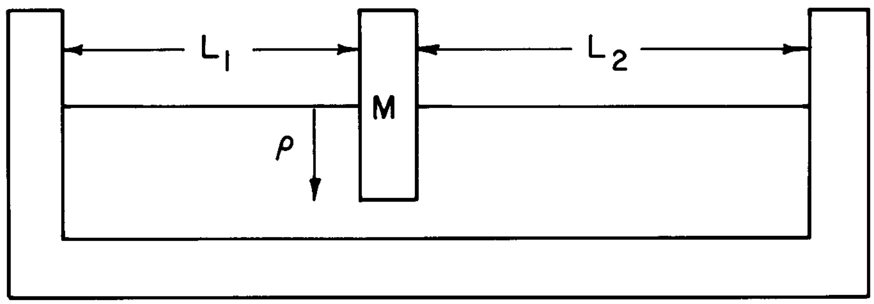


Fig. 2 Schematic drawing of disc and support wires

Consider a solid uniform disc of mass M and radius ρ as shown in Fig.2 above. The axis of the disc is along the horizontal. The disc is supported by two pieces of piano wire of lengths L1 and L2. The disc can rotate about its horizontal axis. The angular position of the disc is denoted by the angle θ. The motion of the disc is obtained from Newton's Third Law:

$$\tau = I\alpha = \frac{d^2\theta}{dt^2}, \tag{1}$$

where α is the angular acceleration and $I = 1/2\, M\rho^2$. Torques are exerted by the wires shown in Fig. 2.

$$\tau_1 + \tau_2 = -K_1\theta - K_2\theta = -K\theta \tag{2}$$

K represents the combined torsional spring constant of the two wires. K1, the torsional spring constant for the left-hand wire is

$$K_1 = G\frac{\pi}{2}r^4\frac{1}{L_1} \tag{3}$$

where r is the radius of the wire and G is the shear modulus of the material from which the wire is made. (See, for example, Mechanics by W. Arthur and S. K. Fenster, p. 513.) Using Eq. 3 for K_1 and K_2, we obtain the total spring constant, K.

$$K = G\frac{\pi}{2}r^4\left(\frac{1}{L_1}+\frac{1}{L_2}\right) \tag{4}$$

In the absence of additional torques the equation of motion of the disc is

$$I\frac{d^2\theta}{dt^2} = -K\theta\,. \tag{5}$$

The above equation is recognized as the equation of motion of a simple harmonic oscillator. The angular frequency is $\omega = \sqrt{K/I}$.

There will be additional torques on the disc due to the various damping forces investigated in the experiments.

Viscous damping

We expect that resistive damping will have a term proportional to $d\theta/dt$, the angular velocity. This is called viscous damping, such as is achieved by placing the disk between the poles of a magnet. The torque due to viscous damping is

$$\tau_{viscous} = R\frac{|\dot{\theta}|^2}{\dot{\theta}} = R\dot{\theta}\,, \tag{6}$$

where is R is a constant with (for viscous damping) units of N·m·s. With viscous damping the equation of motion is

$$I\frac{d^2\theta}{dt^2} + R\frac{d\theta}{dt} + K\theta = 0\,. \tag{7}$$

This equation is the familiar linear, second order differential equation. The solution is of the form $\theta(t) = Ae^{st}$. Solving for s gives

$$s_1, s_2 = -a \pm b = -\left(\frac{R}{2I}\right) \pm \sqrt{\left(\frac{R}{2I}\right)^2 - \left(\frac{K}{I}\right)} \qquad (8)$$

The nature of the solution depends on the sign.

If $b^2 > 0$, then the solution is overdamped: *θ(t)* falls to zero smoothly with no oscillations. The solution is of the form

$$\theta(t) = e^{-at}\left(A_1 e^{bt} + B_1 e^{-bt}\right). \qquad (9)$$

(Note that $a > b$.) The constants A_1 and B_1 are determined from the initial conditions. For $\theta(0) = \theta_o$ and $\dot{\theta}(0) = 0$ (initial angular displacement, no initial angular velocity),

$$\theta(t) = \theta_o e^{-at}\left(\cosh bt + \frac{a}{b}\sinh bt\right) \to \frac{\theta_o}{2}\left(1 + \frac{a}{b}\right)e^{-(a-b)\,t} \qquad (a-b)t >> 1. (10)$$

For b^2 = 0, the solution is critically damped. *θ(t)* will fall to zero in the minimum time without oscillation. There are no longer two distinct solutions to Eq. 8, and the form of the solution is now

$$\theta(t) = \left(A_2 + B_2 t\right)e^{-at} \qquad (11)$$

Again, the constants A_2 and B_2 are determined from the initial conditions. For $\theta(0) = \theta_o$ and $\dot{\theta}(0) = 0$ (initial angular displacement, no initial angular velocity),

$$\theta(t) = \theta_o\left(1 + at\right)e^{-at}. \qquad (12)$$

For critical damping b^2 = 0 implies that $R_{critical} = \sqrt{4\,K\,I}$ and $a = \sqrt{K/I}$.

If b^2 < 0, *b* is a complex number, and we have an oscillatory solution of the form

$$\theta(t) = e^{-at}\left(A_3 e^{ibt} + B_3 e^{-ibt}\right) \qquad (13)$$

Again, the constants A_3 and B_3 are determined from the initial conditions. For $\theta(0)=\theta_o$ and $\dot{\theta}(0)=0$ (initial angular displacement, no initial angular velocity),

$$\theta(t)=\theta_o e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right) \tag{14}$$

Coulomb damping

The motion of the disc can also be damped by kinetic friction, also known as Coulomb damping. For Coulomb damping as long as the disc is moving, the torque is a constant, independent of the magnitude but dependent on the direction of the velocity. The torque always opposes the motion of the disc. Coulomb damping then is described by the torque,

$$\tau_{Coulomb}=C\frac{|\dot{\theta}|}{\dot{\theta}}. \tag{15}$$

where C has units of Nm. The differential equation then becomes

$$I\ddot{\theta}+C\frac{|\dot{\theta}|}{\dot{\theta}}+K\theta=0. \tag{16}$$

There are actually two differential equations here. For $\dot{\theta}>0$ the equation is $I\ddot{\theta}+K\theta=-C$, and for $\dot{\theta}<0$ the equation is $I\ddot{\theta}+K\theta=C$. Both of these equations are easily solved. They are linear, second order inhomogeneous equations. If C were zero, the equations become the equation for a simple harmonic oscillator. We need only to add the particular solution to the homogeneous solution, and the particular solution is the constant $\pm C/K$.

Suppose that we have the usual initial conditions, $\theta(0)=\theta_o$ and $\dot{\theta}(0)=0$. Then initially θ after $t=0$ must decrease and, therefore, $\dot{\theta}<0$. The solution for this case is

$$\theta(t)=C/K+(\theta_o-C/K)\cos\omega t \qquad 0\le t\le\pi/\omega$$

where $\omega=\sqrt{K/I}$. This solution is valid for half a period, or, equivalently, until $\dot{\theta}$ goes through zero and changes sign to $\dot{\theta}>0$. Then the equation becomes $I\ddot{\theta}+K\theta=-C$, and the solution is

$$\theta(t)=-C/K+(\theta_o-3C/K)\cos\omega t \qquad \pi/\omega\le t\le 2\pi/\omega.$$

Note that after a half of a period at $t = 2\pi/\omega$ $\theta(2\pi/\omega) = -\left(\theta_o - 2C/K\right)$. In one period the amplitude of the oscillation has decreased by $\theta(0) - \left|\theta(2\pi/\omega)\right| = 2C/K$. After iterations we obtain

$$\theta(t) = +C/K + (\theta_o - (4n-3)C/K)\cos\omega t \qquad (n-1)\frac{2\pi}{\omega} \le t \le (n-\frac{1}{2})\frac{2\pi}{\omega} \quad n = 1,2,\ldots$$

$$\theta(t) = -C/K + (\theta_o - (4n-1)C/K)\cos\omega t \qquad (n-\frac{1}{2})\frac{2\pi}{\omega} \le t \le n\frac{2\pi}{\omega} \quad n = 1,2,\ldots \tag{17}$$

In every period the amplitude decreases by the constant $4C/K$. In contrast to viscous damping for which the amplitude decreases exponentially, for Coulomb damping the amplitude decreases linearly. At the end of some half period the restoring torque will be less than the frictional torque, i.e. $K\,|\,\theta(t = n\,\pi/\omega)\,| < C$ for some *n*, and the motion will stop. The disc does not return to its equilibrium position, $\theta = 0$

Turbulent damping

Since the disc moves through air, the motion of the disc can be damped by the air turbulence. The damping due to turbulence can be approximated by a power of the angular velocity

$$\tau_{turbulent} = C\frac{|\,\dot\theta\,|^{n+1}}{\dot\theta} \tag{18}$$

where n ≥ 2. The constant C now must have units of N·m·s^n. This gives us a non-linear, second order differential equation,

$$I\ddot\theta + C\frac{|\,\dot\theta\,|^{n+1}}{\dot\theta} + K\theta = 0\,, \tag{19}$$

which, in general, has no closed form solution. The damping term is more phenomenological than fundamental, and a non-integer value for *n* may better describe the phenomenon.

Although we cannot in general obtain a closed form solution for Eq. 19, we can obtain an approximate solution in the situation that the damping is small so that the motion is still oscillatory, albeit with a decaying amplitude. Rearranging Eq. 19 gives

$$\frac{d}{dt}E(t)=\frac{d}{dt}\left(\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}K\theta^2\right)=-C\,|\,\dot{\theta}\,|^{n+1}. \tag{20}$$

The left hand side is recognized as the time derivative of the total energy, the sum of kinetic and potential energy. The total energy decreases at a rate determined by the damping. If we integrate over one period, we obtain an expression for the energy loss in a period,

$$\Delta E=\int_0^T dt\frac{d}{dt}E=-C\int_0^T dt\ \ |\,\dot{\theta}\,|^{n+1}, \tag{21}$$

which without loss of generality we assume begins at t = 0. Now we suppose that over one period the motion is harmonic so that $\theta(t)=\Theta_o\cos\omega t$ and $\dot{\theta}(t)=-\omega\Theta_o\sin\omega t$. Θ_o is the amplitude at the beginning of the period, and $\omega=\sqrt{K/I}$. Our goal is to find the change in the amplitude over one period. We have assumed small damping so we are allowed to assume that the oscillatory frequency is that of the undamped motion. With this assumed form for the solution, the energy loss in one period becomes

$$\Delta E=-C\int_0^T dt\ \ |\,\Theta_o\omega\sin\omega t\,|^{n+1}=-4C\int_0^{T/4} dt\ \left(\Theta_o\omega\right)^{n+1}\sin^{n+1}\omega t, \tag{22}$$

$$\Delta E=-4C\frac{\left(\Theta_o\omega\right)^{n+1}}{\omega}\int_0^{\pi/2} d\xi\ \sin^{n+1}\xi=-4C\frac{\left(\Theta_o\omega\right)^{n+1}}{\omega}\Gamma_n, \tag{23}$$

where $\Gamma_n=\int_0^{\pi/2} d\xi\ \sin^{n+1}\xi$ is just a definite integral. These integrals can easily be evaluated for integer *n*. The first few values are $\Gamma_0=1$, $\Gamma_1=\pi/4$, and $\Gamma_2=2/3$.

Viscous damping has n = 1, and we find the energy loss in a period to be $\Delta E=-\pi\,\omega\,\Theta_o^{\,2}\,R$. Note that the damping constant, $R=-\Delta E/\pi\,\omega\,\Theta_o^{\,2}$. We are then lead to define an equivalent damping coefficient for n ≠ 1 as

$$R_{eq}=\frac{\Delta E}{\pi\,\omega\,\Theta_o^{\,2}}=\frac{4C\left(\omega\,\Theta_o\right)^{n+1}}{\pi\,\omega^2\,\Theta_o^{\,2}}\Gamma_n=\frac{4C}{\pi}\left(\omega\,\Theta_o\right)^{n-1}\Gamma_n, \tag{24}$$

By construction the equivalent (linear) damping torque, $R_{eq}\dot{\theta}$, has the same energy loss per period as the (non-linear) damping torque, $C\,|\,\dot{\theta}\,|^{n+1}/\dot{\theta}$. Assuming an equivalent damping torque, we can define an equivalent attenuation coefficient,

$$a_{eq} = \frac{R_{eq}}{2I} = -\frac{2C}{\pi I}\left(\omega\,\Theta_o\right)^{n-1}\Gamma_n\,. \tag{25}$$

With an equivalent attenuation coefficient we then assume that the amplitude decays in one period exponentially so that $\theta(T) = \Theta_o \exp(-a_{eq}T)$. Then the log decrement can be found as

$$\delta = \ln\left[\frac{\Theta_o}{\theta(T)}\right] = a_{eq}T = \frac{2C}{\pi I}\left(\omega\,\Theta_o\right)^{n-1}\Gamma_n\,\frac{2\pi}{\omega}. \tag{26}$$

For viscous damping we recover $\delta = aT$. From Eq. 26 we note that $\delta \propto \Theta_o^{\,n-1}$. For turbulent damping, n=2, we obtain

$$\delta = \frac{8}{3}\frac{C}{I}\Theta_o\,. \tag{27}$$

This lengthy development leads to the conclusion that for turbulent damping the log decrement is linear with the initial amplitude, Θ_o. A larger initial amplitude has larger velocities and larger damping.

Driven Torsional Oscillator

The equation of motion for the viscous damped, sinusoidally driven torsional oscillator is

$$I\frac{d^2\theta}{dt^2}+R\frac{d\theta}{dt}+K\theta=\lambda K\theta_o\cos\omega t\,. \tag{28}$$

In the above equation θ is the angular displacement of the disc, *I* is the moment of inertia of the disc, *R* is the viscous damping coefficient, and *K* is the torsional spring constant of the wire. The right hand side represents the external torque on the disc through the wire from the sinusoidal drive motor. The amplitude of the drive is θ_o. The external torque includes the spring constant, *K*, and the factor $\lambda = L_1/(L_1+L_2)$ due to the fact that the drive is separated from the disk by a length of wire. The zero of time is chosen so that the torque from the drive is a maximum at t = 0.

Eq. 28 is a linear, second-order non-homogeneous differential equation. The solution of such an equation is the sum of the solution for no driving torque, the transient solution, and the steady state solution due to the presence of the driving torque. The transient solution is

$$\theta_t(t)=Ae^{-at}\cos(\omega_1 t-\phi)\,, \tag{29}$$

where the decay constant and angular frequency are given by

$$a=\left(\frac{R}{2I}\right) \text{ and } \omega_1=\sqrt{\left(\frac{K}{I}\right)-\left(\frac{R}{2I}\right)^2}\,. \tag{30}$$

The undamped, natural angular frequency is

$$\omega_o=\sqrt{\left(\frac{K}{I}\right)}\,, \tag{31}$$

so the damped frequency is equal to

$$a=\left(\frac{R}{2I}\right) \text{ and } \omega_1=\sqrt{\omega_o^2-a^2}\,. \tag{32}$$

The steady state solution will have the frequency of the driving torque. Since in general ω_1 is not the same as ω, before the transient solution dies out, the motion of the disc is the superposition of two oscillations with different frequencies. The superposition of two frequencies produces the phenomenon of "beats."

We search for a steady-state solution of the form (where we will take the real part)

$$\alpha(t) = \alpha_o e^{i\omega t}, \qquad (33)$$

where α_o is a complex constant. Plugging in, we obtain

$$I(i\omega)^2 \alpha_o + R(i\omega)\alpha_o + K\alpha_o = \lambda K\theta_o . \qquad (34)$$

Solving for α_o, we obtain

$$\alpha_o = \frac{\lambda K\theta_o}{K - I\omega^2 + i\omega R} = \frac{\lambda\theta_o \omega_o^2}{(\omega_o^2 - \omega^2) + i\omega\gamma} . \qquad (35)$$

In addition to the undamped natural angular frequency defined in Eq. 31 above, we also introduce the quantity,

$$\gamma = \frac{R}{I} = 2\frac{R}{2I} = 2a . \qquad (36)$$

The complex coefficient, α_o, a function of the driving angular frequency, ω, can be written in polar form as

$$\alpha_o = B(\omega)e^{-i\beta(\omega)} . \qquad (37)$$

where $B(\omega)$ and $\beta(\omega)$ are given by

$$B(\omega) = \frac{\lambda\theta_o \omega_o^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}} \qquad \tan\beta(\omega) = \frac{\omega\gamma}{\omega_o^2 - \omega^2} . \qquad (38)$$

We take the real part of α_o to find the steady state solution to Eq. 28.

$$\theta_s(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right). \tag{39}$$

The full solution for the damped, driven torsional oscillator is the sum of the transient and steady state solutions, Eq. 29 and Eq. 39.

$$\theta(t) = \theta_t(t) + \theta_s(t) = Ae^{-at}\cos\left(\omega_1 t - \phi\right) + B(\omega)\cos\left(\omega t - \beta(\omega)\right). \tag{40}$$

The constants A and ϕ are determined by matching the solution to the initial conditions.

We have assumed that the oscillator is underdamped so that the oscillatory solution, Eq. 29, describes the transient. We could, in principle, also use the critically damped and over-damped solutions here, but they die out in one period or less and the steady state becomes the only solution.

B. Beats

Two oscillators of equal amplitude and of similar, but not identical, frequencies generate waves. The resultant wave at a given point in space is then the superposition of two frequencies. Then with a judicious choice of phase of the two independent oscillators, we obtain

$$A\sin(\omega t) - A\sin(\omega_o t) \rightarrow 2A\sin\left(\frac{\omega - \omega_o}{2}t\right)\cos\left(\frac{\omega + \omega_o}{2}t\right). \tag{41}$$

The difference in the frequencies is small for $\omega \approx \omega_o$, so $\frac{\omega + \omega_o}{2} \approx \omega_o$. The perception is of a sound with a single frequency but with an unsteady amplitude. It is possible to demonstrate the same phenomenon with the torsional oscillator with a judicious choice of frequencies and initial conditions.

Suppose we hold the disc at its equilibrium position with the drive motor running. Since the disc is at its equilibrium position and is at rest, both $\theta = 0$ and $\dot{\theta} = 0$. If we release the disc when the drive is exerting its maximum torque, we impart an initial angular acceleration. Consider the superposition of the transient and steady state motion, Eq. 40, for an oscillator with small damping and for times for which the transient has not yet decayed away. Then the exponential function in Eq. 16, $\exp(-a\,t) = \exp(-\gamma\,t/2)$, is close to one. We can also substitute ω_o for ω_1. With these conditions the motion of the disc, Eq. 40, becomes, using as initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 0$.

$$\begin{aligned} \theta(t) &= -B\cos(\beta)\cos(\omega_o\,t) - B\omega\sin(\beta)\sin(\omega_o\,t)/\omega_o + B\cos(\omega t - \beta) \\ \dot{\theta}(t) &= B\,\omega_o\cos(\beta)\sin(\omega_o\,t) - B\omega\,\sin(\beta)\cos(\omega_o\,t) - B\omega\sin(\omega t - \beta) \end{aligned} \quad .(42)$$

If we are close to resonance we can set $\omega\,/\omega_o = 1$ and $\beta = \pi/2$, we obtain

$$\begin{aligned} \theta(t) &\approx -B\sin(\omega_o\,t) + B\cos\left(\omega t - \frac{\pi}{2}\right) \\ &= -B\sin(\omega_o\,t) + B\sin(\omega t) \\ &= 2B\sin\left(\frac{\omega_o - \omega}{2}t\right)\cos\left(\frac{\omega_o + \omega}{2}t\right) \end{aligned} \qquad (43)$$

Recall that the amplitude B depends on ω and ω_o, so that

$$\theta(t) \approx 2\frac{\lambda\theta_o\omega_o^2}{\left(\omega_o^2 - \omega^2\right)}\sin\left(\frac{\omega_o - \omega}{2}t\right)\cos\left(\frac{\omega_o + \omega}{2}t\right) \qquad (44)$$

For $\omega \approx \omega_o$ we see in addition to the sinusoidal oscillation at ω, a modulation of the amplitude at the beat (angular) frequency $|\omega_o - \omega|/2$. Note that in many texts the beat (angular) frequency is defined as $|\omega_o - \omega|$. It is also possible to get convincing beats by just turning on the drive motor with the disc at rest.

C. Resonance

If we assume that the transient solution has died away, the motion of the disc is given by

$$\theta(t) = \frac{\lambda\theta_o\,\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}\cos\left(\omega t - \beta(\omega)\right) \qquad t >> 1/\gamma\,. \tag{45}$$

In the high frequency limit $\omega >> \omega_o$ and $\omega >> 1/\gamma$ Eq. 45 gives

$$\theta(t) = \frac{\lambda\theta_o\,\omega_o^2}{\omega^2}\cos(\omega t - \pi)\,. \tag{46}$$

The amplitude decreases as $1/\omega^2$ as shown in the figure above. In the low frequency limit, $\omega \to 0$, the amplitude approaches $\lambda\theta_o$. For ω close to ω_o and with sufficiently small damping that the term, $\gamma^2\omega^2$, in the denominator can be neglected, we find the amplitude on either side of ω_o falls off as $1/\left(\omega_o^2 - \omega^2\right)$. Very close to resonance, $\omega \approx \omega_o$, the $\gamma^2\omega^2$ term in the denominator cannot be neglected, and its magnitude determines the "line-width" of the resonance, i.e. the range of frequencies for which the amplitude is large. The width of the resonance depends strongly on the damping constant, as shown in the figure below.

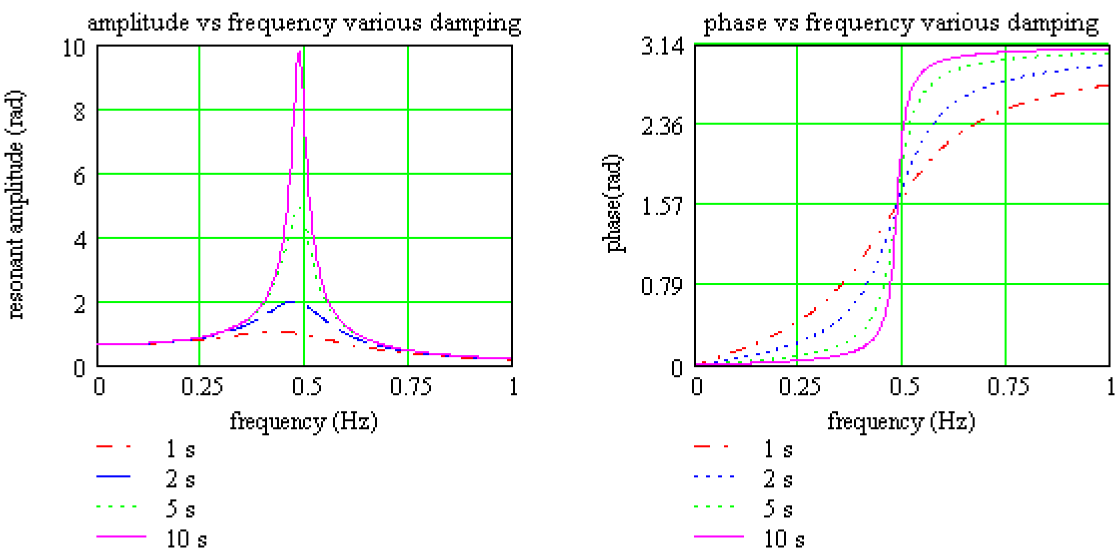


Fig. 1 Amplitude and phase of damped, torsional oscillator versus frequency (calculation)

We might note that the maximum amplitude of oscillation does not fall precisely at resonance, which we have defined as $\omega = \omega_o$. This is a very small point, but perhaps worth noting. The maximum amplitude is given by the (angular) frequency at which $dB/d\omega = 0$. With $B(\omega)$ from Eq. 38 after a little algebra we find that

$$\omega_{\max}^2 = \omega_o^2 - \gamma^2/2 . \tag{47}$$

<u>Data Analysis</u>

A. Quality factor and log decrement

There are many equivalent ways to discuss the motion of the oscillator. Recall that the frequency of oscillation for under damped oscillatory motion is

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{K}{I}\right) - \left(\frac{R}{2I}\right)^2} . \tag{48}$$

The natural (undamped) frequency is $f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{K}{I}\right)}$.

If the motion is oscillatory and the damping is not large, there are two auxiliary parameters used to measure the rate at which the oscillations of a system are damped out. One parameter is the logarithmic decrement, δ, and the other is the quality factor, *Q*. The logarithmic decrement, δ, is defined by

$$\delta = \ln\left(\frac{\theta(t_{\max})}{\theta(t_{\max} + T_1)}\right) = \ln\left(\frac{e^{-at_{\max}}}{e^{-a(t_{\max}+T_1)}}\right) = aT_1 . \tag{49}$$

where t_{max} is the time when θ is at its maximum value and T_1=period. Note that for viscous damping, since *a* and T_1 are constants, δ is the same from initial to late times in the motion.

The *Q* of the circuit, or quality factor, is defined as

$$Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}} .$$

The *Q* can be related to the logarithmic decrement. (Recall for the RLC circuit $Q = \frac{\omega_1 L}{R} \rightarrow \frac{\omega_1 I}{R}$.)

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi}{a}\frac{\omega_1}{2\pi} = \frac{\pi}{a}\frac{1}{T_1} = \frac{\pi}{\delta} \tag{50}$$

For small R, the logarithmic decrement is small and therefore *Q* is large.

The amplitude of the oscillation decays exponentially as e^{-at}. From the definitions of δ and *Q*, we also find that the amplitude decays exponentially as $\exp(-\pi t / Q T_1)$. Thus the amplitude decays by a factor of $1/e$ in a time $t_{(1/e)} = Q T_1 / \pi$. Q/π is the number of cycles of damped motion needed to reduce the amplitude to $1/e$ of its original value. This definition is equivalent to the definition of *Q* in terms of energy loss above.

There is a third definition of Q which is appropriate for frequencies near resonance, as in part C above. We have that

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi}{a}\frac{\omega_1}{2\pi} = \frac{\pi}{a}\frac{1}{T} = \frac{\pi}{\delta} \tag{51}$$

For small damping $\omega_1 \approx \omega_o$ and using $\gamma = 2a$, we obtain an alternative interpretation of Q.

$$Q \approx \frac{\omega_o}{\gamma}\,. \tag{52}$$

For large damping the resonance line is far from symmetric, and Eq. 52 is not accurate.

B. Fourier analysis

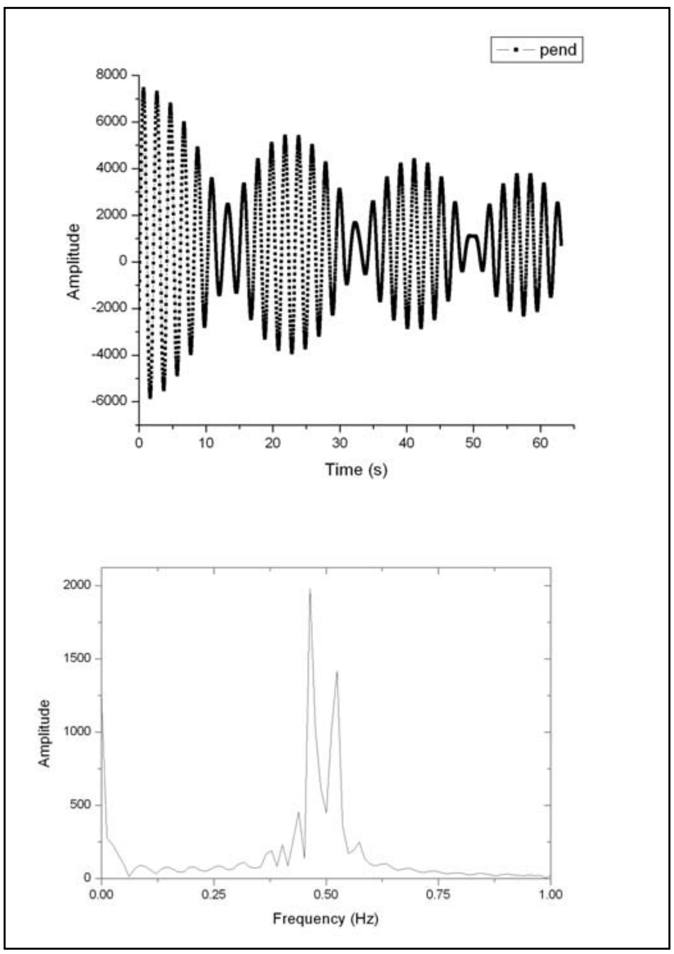
Since the motion of the torsional oscillator is periodic in time – i.e. harmonic, Fourier analysis (FFT) can be a very useful tool for investigating the various frequency components of the oscillator. The Fourier transform decomposes a function in the time domain into its frequency components in the frequency domain. By carrying out a Fourier transform on your data, you will be able to determine which frequencies are present in your system. For example, when investigating the phenomenon of beats in the torsional pendulum data, two distinct frequencies should be observable, one that is associated with the shorter-period oscillation and one associated with the longer-period oscillation. In order to carry out high-quality Fourier transforms of the torsional pendulum data, you should take stable, steady-state data for quite a long time, otherwise frequency artifacts can be introduced that aren't really there. The Fourier transform is defined in the following way:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{i\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{-i\omega t} d\omega$$

where $x(t)$ is the original function in the time domain and $X(\omega)$ is the corresponding function in the frequency domain.

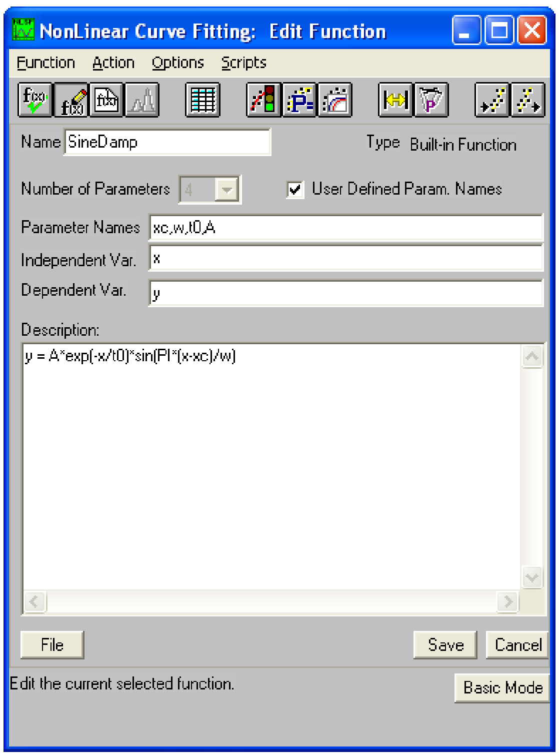
In Origin, Fourier transforms are implemented by going to the "Analysis" menu and clicking on FFT. There are a range of different parameters you can choose, though the defaults work quite well. Below is a sample set of data and its Fourier transform. The large DC component comes from the fact that the last point in the data set is not equal to the first point in the data set.

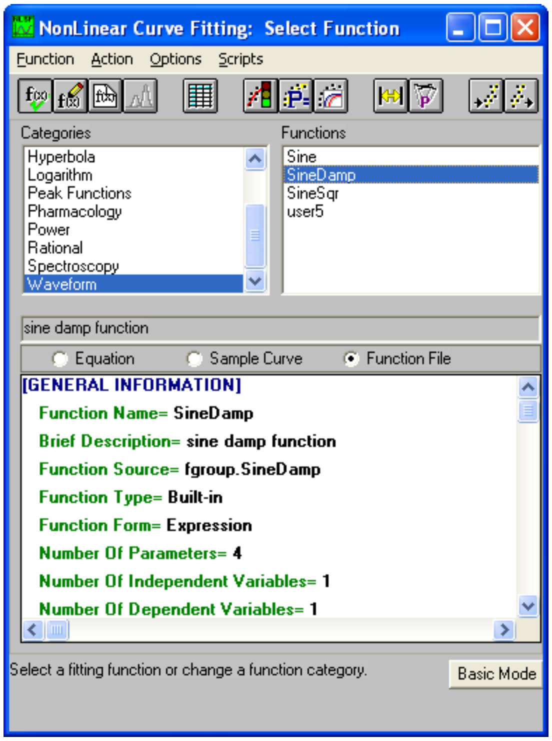


As one can see there are two peaks: one corresponding to the higher frequency/shorter period oscillation, the other corresponding to the lower frequency/longer-period oscillation. It is sometimes also possible to see higher frequency components e.g. a $2\omega, 4\omega, \ldots$. These are caused by the departure of the mechanical linkage of the drive motor from being a perfect sinusoid. Fourier analysis can be a very useful tool for understanding your data and should be used in your report.

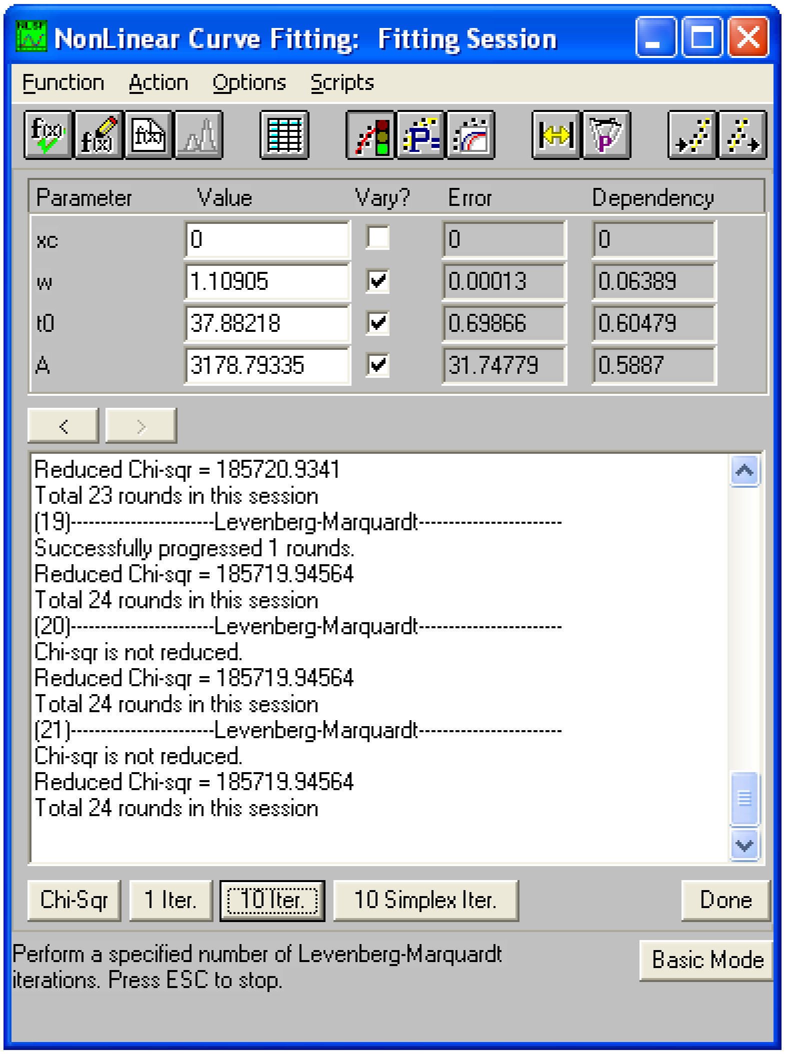
C. Fitting functions

Origin, in addition to doing Fourier transforms, can also fit a function to your data. The data to fit is the data from magnetic damping, though in principle any data can be fit. The theory suggests that the best type of fit to do would be a sinusoidal wave whose amplitude is damped exponentially. Before fitting your data, it is important that you make sure the data oscillates about zero. In origin, this type of fit can be found in the "Analysis" menu under "Non-linear curve fit." This will bring up a dialog box. Click the "Function" menu and select "Select." Under "Categories" select "Wavefunction" and under "Functions" select "Sinedamp." By selecting the second button in the top row, you can see how the function is defined (see figure below for an example).

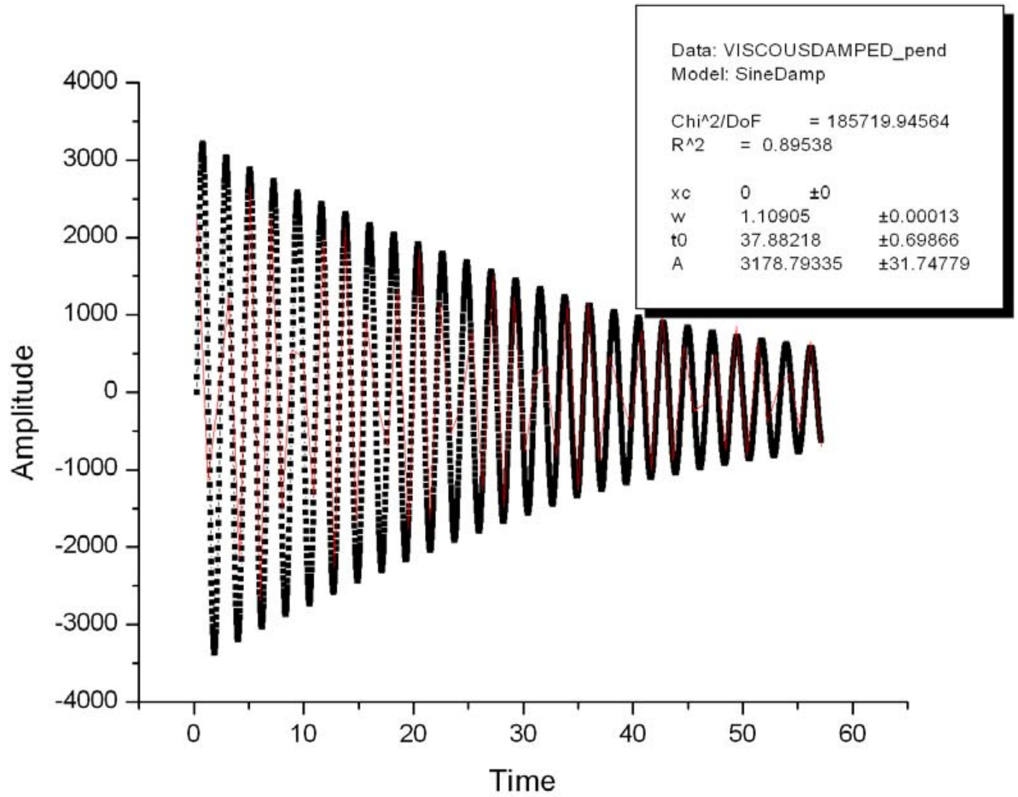




There are four parameters that the function will fit: xc, w, t0, and A. xc is the y-value of your function at x=0. A is the initial amplitude of the function (the overall highest or lowest point). P1 is defined to be 2π. w is the angular frequency (so w determines the number of seconds in one period). t0 determines how long it takes for the amplitude to decay to 1/e of its original value. To fit your function, click on the green light button.



A dialog box will pop up asking if you wish to fit the current data set. Click yes. You need to input your guesses for the four fitting parameters depending on what your data looks like. You may get an error suggesting you set xc as fixed. Go ahead and do this, since the value of xc shouldn't change. To fit the data, click on 10 Iter. This will do 10 iterations of fitting and then show you the final results. Continue fitting until the error on each parameter does not change or, alternately, until Chi-sqr is no longer reduced. Chi-squared is a general measure of how good the fit is. The lower the chi-squared, the better the fit, in general. In the case of this fitting, it is not such a useful measure of “goodness” of the fit. A better measure is how small the error on each parameter is and the value of R^2, which you will see once the fit is over. R^2 should be as close to 1 as possible.



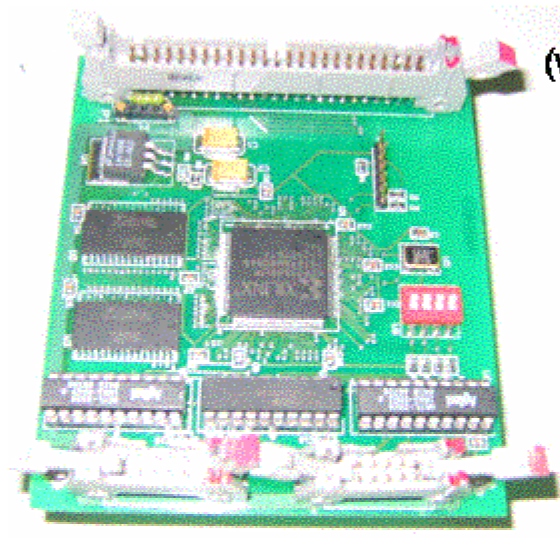
In the graph above is an example of an origin fit to magnetic damping data. You can see that R^2 =0.89538. This means that the fit is rather good, even though chi-squared is so large. From this fit, you can get a good feel for the time constant and the frequency of the sine wave.

Appendix II
Buffer Board Switch Positions

Buffer board switch positions

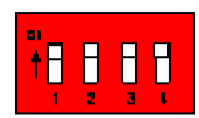
Time Base Settings
(from pull-down menu)

1. 5 Hz
2. 10 Hz
3. 20 Hz
4. 25 Hz
5. 40 Hz
6. 50 Hz
7. 100 Hz
8. 200 Hz



ON is up position
(with this board orientation)

	1	2	3	4
5	off	off	off	on
10	on	off	off	on
20	off	on	off	on
25	on	off	on	off
40	on	on	off	on
50	on	on	on	on
100	on	on	on	off
200	on	on	off	off



50 Hz selected